Convex-roof extended negativity as an entanglement measure for bipartite quantum systems

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August, 2003

Contents

References

- **W. K. Wootters**
  - *PRL, V. 80, 2245 (1998)*
  - *Entanglement of formation of an arbitrary state of two qubits*
- **G. Vidal**
  - *J. Mod. Opt., V. 47, 355 (2000)*
  - *Entanglement monotones*
- **W. Dür, J. I. Cirac, M. Lewenstein & D. Bruß**
  - *PRA, V. 61, 062313 (2000)*
  - *Distillability and partial transposition in bipartite systems*
- **B. M. Terhal & K. G. H. Vollbrecht**
  - *PRL, V. 85, 2625 (2000)*
  - *Entanglement of formation for isotropic states*
- **K. G. H. Vollbrecht & R. F. Werner**
  - *PRA, V. 64, 062307 (2001)*
  - *Entanglement measures under symmetry*
- **G. Vidal & R. F. Werner**
  - *PRA, V. 65, 032314 (2002)*
  - *Computable measure of entanglement*
- **P. Rungta & C. M. Caves**
  - *PRA, V. 67, 012307 (2003)*
  - *Concurrence-based entanglement measures for isotropic states*

Bipartite entanglement

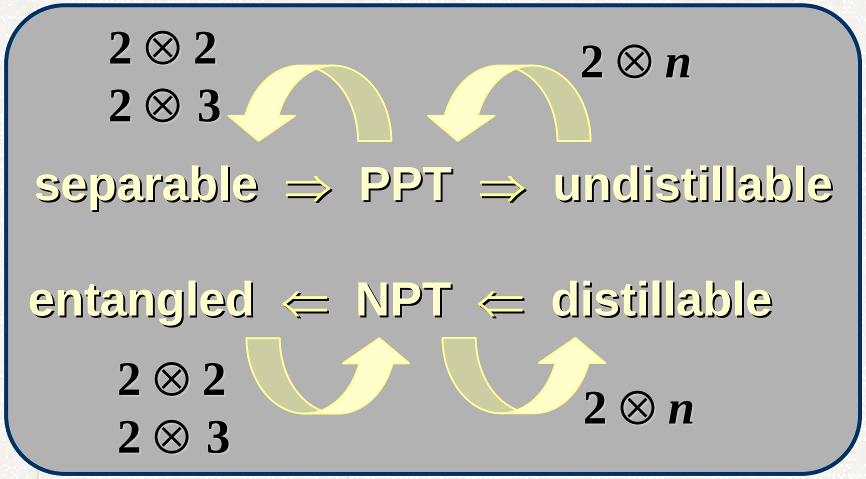
- *Pure state* $|\Psi\rangle\langle\Psi|$
  - $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \Leftrightarrow$ *separable*
  - $|\Psi\rangle\langle\Psi| = |\Psi_A\rangle\langle\Psi_A| \otimes |\Psi_B\rangle\langle\Psi_B| \Leftrightarrow$ *separable*
  - *Not separable* $\Leftrightarrow$ *entangled*
- *Mixed state* ρ
  - $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i \; (p_i \geq 0, \; \sum_i p_i = 1)$ $\Leftrightarrow$ *separable*
  - *Not separable* $\Leftrightarrow$ *entangled*

Quantum information and entanglement

- *Applications of entanglement*
  - *Teleportation*
  - *Key Distribution*
  - *Dense Coding*
  - *Parallel Computation*
- *Problems for entanglement in quantum information science*
  - *Separability criteria*
    - *To find a method to determine whether a given state in any dimensional quantum system is separable or not.*
  - *Entanglement measures*
    - *To define the best measure quantifying an amount of entanglement of a given state.*

Positive partial transposition (PPT)

- *Separable* $\Rightarrow$ *PPT* $\Rightarrow$ *undistillable*
  - *Distillable* $\Rightarrow$ *Negative Partial Transposition (NPT)* $\Rightarrow$ *entangled*
- $\mathbf{2} \otimes \mathbf{2}$ *or* $\mathbf{2} \otimes \mathbf{3}$ *case*
  - *PPT* $\Leftrightarrow$ *separable*
- $\mathbf{2} \otimes \boldsymbol{n}$ *case*
  - *PPT* $\Leftrightarrow$ *undistillable*



Pure states in $d \otimes d$ quantum systems

- $|\Psi\rangle = \sum_i \mu_i^{1/2} | a_i b_i\rangle = (U_A \otimes V_B)|\Phi\rangle$
  - $|\Phi\rangle = \sum_i \mu_i^{1/2} | i\, i\rangle$
- *Partial transposition of* $|\Phi\rangle\langle\Phi|$
  - $|\Psi_{ij}^{\pm}\rangle = (| i\, j \rangle \pm | j\, i \rangle) / 2^{1/2}$
  - $|\Phi\rangle\langle\Phi|^{T_B} = \sum_i \mu_i | i\, i\rangle\langle i\, i| + \sum_{i<j} (\mu_i \mu_j)^{1/2} |\Psi_{ij}^{+}\rangle\langle\Psi_{ij}^{+}| - \sum_{i<j} (\mu_i \mu_j)^{1/2} |\Psi_{ij}^{-}\rangle\langle\Psi_{ij}^{-}|$
  - $N_p(\mu) \equiv N_p(|\Psi\rangle) = N_p(|\Phi\rangle) = 2 \sum_{i<j} (\mu_i \mu_j)^{1/2} / (d-1)$
    - $\mu = (\mu_0^{1/2}, \mu_1^{1/2}, \ldots, \mu_{d-1}^{1/2})$: *the Schmidt vector of* $|\Psi\rangle$
    - $N_p((1, 1, \ldots, 1)/d^{1/2}) = 1$
- *Negativity for pure states*
  - $N(\rho) = (\| \rho^{T_B} \| - 1)/(d - 1)$
  - $N_p(\mu) = (\| |\Psi\rangle\langle\Psi|^{T_B} \| - 1)/(d - 1) = N(|\Psi\rangle\langle\Psi|)$

Convex-roof extended negativity (CREN)

- *Mixed state* ρ *in* $\mathbf{d} \otimes \mathbf{d}$ *quantum systems*
  - *Convex-roof extension of* N_p
  - $N_m(\rho) \equiv \min \sum_k p_k N_p(\mu_k)$
    The minimum is taken over all possible decomposition of $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$.
    - μ_k: *the Schmidt vector of* $|\Psi_k\rangle$
- N_m *& the original negativity* N
  - $N_m(\rho) \geq N(\rho) = (\,\|\rho^{T_B}\| - 1\,)/(d-1)$
    - *Convexity of* N
    - *Triangle inequality of* $\|\cdot\|$
- *Pure states in* $2 \otimes 2$ *quantum system*
  - $N_p(\mu) = N_p(|\Psi\rangle) = 2\,(\mu_0\,\mu_1)^{1/2} = |\langle\Psi|\sigma_y\otimes\sigma_y|\Psi^*\rangle| = C(|\Psi\rangle)$
    - $C(|\Psi\rangle)$: *concurrence of* $|\Psi\rangle$
  - $N_m(\rho) = C(\rho)$

Separability and bound entanglement

- *Bound entanglement (BE)*
  - *Undistillable entanglement*
- $N(BE) \geq 0$, "=" $\Leftrightarrow$ *(BE) = (PPT entangled)*
- $N_m(BE) > 0$
- ρ: *separable* $\Leftrightarrow$ $N_m(\rho) = 0$
  - ρ: *separable* $\Rightarrow$ $N(\rho) = 0$

Entanglement monotones

- *E: entanglement monotone*
  - $\rho \xrightarrow{\text{LOCC}} \rho' \Rightarrow \boldsymbol{E}(\rho) \geq \boldsymbol{E}(\rho')$
- *G. Vidal*
  - *J. Mod. Opt., V. 47, 355 (2000)*
  - *$\boldsymbol{E}$: a pure-state measure of entanglement*
    - *Define a function* f *by* $f(\mathrm{tr}_B|\Psi\rangle\langle\Psi|) \equiv \boldsymbol{E}(|\Psi\rangle)$.
    - *Invariance under unitary operations*
      - $f(U\rho U^\dagger) = f(\rho)$ *for any unitary* $\boldsymbol{U}$.
    - *Concavity*
      - $f(\lambda\rho_1 + (1-\lambda)\rho_2) \geq \lambda f(\rho_1) + (1-\lambda) f(\rho_2)$ *for any* ρ_1, ρ_2 *and* $\lambda \in [0,1]$.
  - *The convex-roof extension of E is an entanglement monotone.*

$$f(\rho) = [\mathrm{tr}(\rho^{1/2})^2 - 1]/(d-1)$$

- $f(\rho) \equiv N_p(\mu) = 2\sum_{i<j}(\mu_i\,\mu_j)^{1/2}/(d-1) = \left[\left(\sum_i \mu_i^{1/2}\right)^2 - 1\right]/(d-1)$
  - μ *: the vector with entries consisting of eigenvalues of* ρ.
- *N_m is an entanglement monotone.*

Entanglement measures under symmetry

- *K. G. H. Vollbrecht & R. F. Werner*
- *PRA, V. 64, 062307 (2001)*
- *Preliminaries*
  - K : a compact convex set
  - M : a subset of K
  - G : a compact group of symmetries acting on K by $(U, x) \mapsto UxU^\dagger$
    - $UMU^\dagger \subset M$
  - A function $f : M \to \mathbb{R} \cup \{+\infty\}$
    - For $s \in M$, $U \in G$, $f(UsU^\dagger) = f(s)$
  - The projection $\mathbf{P}: K \to K$
    - $\mathbf{P}x = \int dU\, UxU^\dagger$
    - $\mathbf{P}K$: the set of all G-invariant elements in K
  - The function $\varepsilon : \mathbf{P}K \to \mathbb{R} \cup \{+\infty\}$
    - $\varepsilon(x) = \min\{f(s) : s \in M, \mathbf{P}s = x\}$
- $\mathrm{co}\, f(x) = \mathrm{co}\,\varepsilon(x), \; x \in \mathbf{P}K$

Isotropic states in $d \otimes d$ systems

- $\rho_F = (1-F)/(d^2-1)\,(\mathrm{I} - |\Psi^+\rangle\langle\Psi^+|) + F\,|\Psi^+\rangle\langle\Psi^+|$
  - $|\Psi^+\rangle = (1/d^{1/2})\sum_i |i\,i\rangle$
- $0 \le F = \langle\Psi^+|\rho_F|\Psi^+\rangle \le 1$
- ρ_F : *separable* $\Leftrightarrow$ ρ_F : *PPT* $\Leftrightarrow$ $0 \le F \le 1/d$
- *Negativity*
  - $N_m(\rho_F) = (Fd-1)/(d-1) = N(\rho_F)$

$T(\rho) = \int_{U(d)} dU\,(U \otimes U^*)\,\rho\,(U \otimes U^*)^\dagger$

dU: normalized Haar measure on $U(d)$

$T(\rho_F) = \rho_F$

Werner states in $d \otimes d$ systems

- $\square_W = \alpha I + \beta \sum_{i,j} | i\, j \rangle\langle j\, i |$
  - $(\alpha d + \beta) d = 1$
  - $W = 1/2 - \mathrm{tr}(\square_W \sum_{i,j} | i\, j\rangle\langle j\, i|)/2$
  - $\square_W = 2(1-W)/d(d+1) \left(\sum_i |i\, i\rangle\langle i\, i| + \sum_{i<j} |\Psi_{ij}^{+}\rangle\langle\Psi_{ij}^{+}|\right)$
    $+ 2W/d(d-1) \sum_{i<j} |\Psi_{ij}^{-}\rangle\langle\Psi_{ij}^{-}|$
  - $W = \mathrm{tr}(\square_W \sum_{i<j} |\Psi_{ij}^{-}\rangle\langle\Psi_{ij}^{-}|)$
- $\square_W$: *separable* $\Leftrightarrow$ $\square_W$: *PPT* $\Leftrightarrow$ $0 \le W \le 1/2$
- *Negativity*
  - $N_m(\square_W) = (2W-1)/(d-1) \ge 2(2W-1)/d(d-1) = N(\square_W)$

$T(\rho) = \int_{U(d)} dU\, (U \otimes U)\, \rho\, (U \otimes U)^{\dagger}$

dU: normalized Haar measure on $U(d)$

$T(\square_W) = \square_W$

Conclusions

- *The mathematical expressions for the CREN are less complicated than those of other convex-roof extended measures.*
- *The CREN can recognize the difference between separable states and bound entangled states.*
- *The CREN is a good candidate for the entanglement measures in bipartite quantum systems.*

Further studies

- *Some other states*
  - *Bound entangled states*
- *Comparisons between* $N_{\mathbf{m}}$ *and other entanglement measures*
- $N_{\mathbf{m}}^2(\rho) \equiv \min \sum_k p_k N_{\mathbf{p}}^2(|\Psi_k\rangle)$